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Non-zero solutions for a class of generalized variational inequalities in reflexive Banach spaces[☆,☆☆]

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Abstract

In this paper, we study the existence of nonzero solutions for a class of generalized variational inequalities involving set-contractive mappings by using the fixed point index approach in reflexive Banach spaces. Under some suitable assumptions, we show some new existence theorems of nonzero solutions for this class of generalized variational inequalities in reflexive Banach spaces.

Keywords: Variational inequality; Fixed point index; Nonzero solution; Set-contractive mapping

1. Introduction and preliminaries

Variational inequality theory a is very powerful tool of the current mathematical technology. In recent years, variational inequalities have been generalized and extended in many different directions using novel and innovative techniques to study wider classes of unrelated problems arising in optimization and control, economic and finance, transportation and electrical networks, operations research and engineering sciences in a general and unified framework (see, for example, [1–3] and the references therein).

It is well known that the fixed point theory plays an important role in the study of variational inequalities and related fields. Recently, some nonzero fixed point theory theorems in Hilbert spaces which are special cases of variational inequalities and some applications to existence of positive solutions for some integral equations were given by Lan [4], Lan and Webb [5].

On the other hand, motivated by the paper [6], Zhu [7] studied a system of variational inequalities involving the linear operators in reflexive Banach spaces by using the coincidence degree theory due to Mawhin [8]. Some existence results of positive solutions for this system of variational inequalities in reflexive Banach spaces were proved. Recently, Lai et al. [9] obtained some existence theorems of nonzero solutions for a class of variational inequalities

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by the fixed point index approach for set-valued mappings in finite-dimensional spaces and reflexive Banach spaces, respectively. Under some different conditions, Lai and Zhu [10] also got some new existence theorems of nonzero solutions for this class of variational inequalities in these spaces.

Let X be a real Banach space, X^* its dual and $(\cdot,\cdot)$ the duality pairing between X^* and X. Suppose that K is a nonempty closed convex subset of X. Motivated and inspired by recent research works in this field, in this paper, we study the existence of nonzero solutions for a class of generalized variational inequalities involving nonlinear mappings as follows:

Find $u \in K$, $u \neq 0$ such that

$$(Au, v-u) \geq (g(u), v-u), \quad \forall v \in K, \tag{1.1}$$

where $A, g : K \to X^*$ are two nonlinear mappings.

By using the fixed point index approach, we show some new existence theorems of nonzero solutions for the generalized variational inequality (1.1) in reflexive Banach spaces under some suitable assumptions.

An operator $A : K \to X^*$ is said to be continuous on finite dimensional subspaces of X if the restriction of A to the intersection of K with any finite dimensional subspaces of X is weakly continuous.

Let X and X_1 be two real Banach spaces and $D \subset X$. A continuous mapping $A : D \to X_1$ is said to be k-set-contractive on D if there exists a constant $k \geq 0$ such that $\alpha(A(S)) \leq k\alpha(S)$ for all bounded $S \subset D$, where α is the Kuratowski measure of noncompactness. If $k < 1$ we shall say that A is strict set-contractive. A continuous mapping $A : D \to X_1$ is said to be condensing if $\alpha(A(S)) < \alpha(S)$, whenever $S \subset D$ is bounded and $\alpha(S) > 0$.

Let X be a real Banach space and K be a nonempty closed convex and U be open and bounded of X with $U \cap K \neq \emptyset$. The closure and boundary of U relative to K are denoted by $\overline{U}_K$ and ∂U_K, respectively. Suppose that $A : \overline{U}_K \to K$ is strict set-contractive and $x \neq A(x)$ for $x \in \partial U_K$. Then it is well known that the fixed point index $i_K(A, U)$ is well defined (see [11]).

Lemma 1.1 ([11]). *Let K be a nonempty closed convex subset of a real Banach space X and U be an open bounded subset of X. Suppose that $A : \overline{U}_K \to K$ is strict set-contractive and $x \neq A(x)$ for $x \in \partial U_K$. Then the fixed point index $i_K(A, U)$ has the following properties:*

(i) *For any mapping $\hat{x}_0$ with constant value x_0, if $x_0 \in U_K$, then $i_K(\hat{x}_0, U) = 1$;*
(ii) *$i_K(A, U_1 \cup U_2) = i_K(A, U_1) + i_K(A, U_2)$, whenever U_1 and U_2 are disjoint open subsets of X such that $x \neq Ax$ for $x \in \partial((U_1)_K) \cup \partial((U_2)_K)$;*
(iii) *Let $H : [0,1] \times \overline{U}_K \to K$ be a continuous, bounded mapping and for each $t \in [0,1]$, $H(t,\cdot)$ be a strict set-contractive mapping. Suppose that $H(t,x)$ is uniformly continuous at t for all $x \in \overline{U_K}$ and for all $(t,x) \in [0,1] \times \partial(U_K)$, $x \neq H(t,x)$, then $i_K(H(1,\cdot), U) = i_K(H(0,\cdot), U)$;*
(iv) *If $i_K(A, U) \neq 0$, then T has a fixed point in U_K.*

2. Main results

Let X be a real reflexive Banach space and K a nonempty closed convex subset of X. Suppose that the mapping $A : K \to X^*$ is continuous on finite dimensional subspaces such that

$$(Au - Av, u - v) \geq \gamma |u - v|^2, \tag{2.1}$$

where $\gamma > 0$ is a constant. It is easy to see that the condition (2.1) implies that A is strictly monotonic and coercive. Hence, for any given $w \in X^*$, the following variational inequality

$$u \in K : (Au, v-u) \geq (w, v-u), \quad \forall v \in K$$

has a unique solution $u \in K$ (see [2]). Now we define a mapping $K_A : X^* \to K$ as follows:

$$K_A(\omega) = u, \quad \forall \omega \in X^*. \tag{2.2}$$

It follows that

$$\|K_A(\omega_1) - K_A(\omega_2)\| \leq \frac{1}{\gamma}\|\omega_1 - \omega_2\|_{X^*}, \tag{2.3}$$

and so K_A is continuous and bounded.

Theorem 2.1. *Let X be a real reflexive Banach space and K a closed convex cone of X. Suppose that $A : X \to X^*$ is continuous on finite dimensional subspaces and bounded which satisfies the condition (2.1) with $A(0) = 0$ and $g : K \to X^*$ is β-set-contractive, where $\frac{\beta}{\gamma} < 1$. Assume that*

(i) $\liminf_{\|u_n\|\to 0} \frac{(gu_n,u_n)}{\|u_n\|^2} < \gamma (u_n \in K)$;
(ii) *there exist $x_0 \in K$ and a constant $q > 0$ such that*

$$(A(u), x_0) < (g(u), x_0)$$

for all $u \in K$ with $\|u\| > q$.

Then generalized variational inequality (1.1) has a nonzero solution.

Proof. For the mappings A and g, we define a new mapping $K_{Ag} : K \to K$ as follows:

$$K_{Ag}(u) = K_A(g(u)), \quad \forall u \in K, \tag{2.4}$$

where K_A is defined by (2.2). It follows from (2.3) that K_{Ag} is $\frac{\beta}{\gamma}$-set-contractive. Let $K^r = \{x \in K, \|x\| < r\}$. Next we shall verify that $i_K(K_{Ag}, K^r) = 1$ for small enough r and $i_K(K_{Ag}, K^R) = 0$ for large enough R.

First, define a mapping by $H : [0, 1] \times \overline{K^r} \to K$, $H(t, u) = tK_A(g(u))$. Clearly, H is continuous bounded in $[0, 1] \times \overline{K^r}$ and for each $t \in [0, 1]$, $H(t, \cdot)$ is a strict set-contractive mapping. It is easy to see that $H(t, x)$ is uniformly continuous at t for all $x \in \overline{K^r}$. Next we prove that there exists small enough $r > 0$ such that $u \neq H(t, u)$ for all $t \in [0, 1]$, $u \in \partial(K^r)$. Otherwise, there exist two sequences $\{t_n\}$ and $\{u_n\}$ with $t_n \in (0, 1]$, $u_n \in \partial(K^r)$, and $\|u_n\| \to 0$ such that

$$u_n = H(t_n, u_n) = t_n K_{Ag}(u_n)$$

or

$$\frac{u_n}{t_n} = K_{Ag}(u_n).$$

Hence, we have

$$\left(A\left(\frac{u_n}{t_n}\right), v - \frac{u_n}{t_n}\right) \geq \left(g(u_n), v - \frac{u_n}{t_n}\right), \quad \forall v \in K. \tag{2.5}$$

Letting $v = 0$, it follows from (2.1) and (2.5) that

$$\gamma \leq \frac{(A(\frac{u_n}{t_n}), \frac{u_n}{t_n})}{\|\frac{u_n}{t_n}\|^2} \leq \frac{(g(u_n), \frac{u_n}{t_n})}{\|\frac{u_n}{t_n}\|^2} \leq \frac{(g(u_n), u_n)}{\|u_n\|^2}. \tag{2.6}$$

Thus

$$\liminf_{\|u_n\|\to 0} \frac{(gu_n, u_n)}{\|u_n\|^2} \geq \gamma,$$

which contradicts condition (i). Therefore,

$$i_K(K_{Ag}, K^r) = i_K(H(1, \cdot), K^r) = i_K(H(0, \cdot), K^r) = i_K(0, K^r) = 1. \tag{2.7}$$

Secondly, we shall verify that $i_K(K_{Ag}, K^R) = 0$ for large enough R. In fact, since g is β-set-contractive and A is bounded, there exist two constants $L > 0$ and $M > 0$ such that $\|g(u)\| \leq L$ and $\|A(u)\| \leq M$ whenever $\|u\| \leq q$, $u \in K$. Let N be large enough and $f \in X^*$ such that

$$M\|x_0\| + L\|x_0\| < -N(f, x_0). \tag{2.8}$$

Define a mapping $H : [0, 1] \times \overline{K^R} \to K$ as follows:

$$H(t, u) = K_A(g(u) - tNf), \quad \forall (t, u) \in [0, 1] \times \overline{K^R}.$$

Then $H(t,u)$ is continuous and bounded in $[0,1]\times\overline{K^R}$ and for each $t\in[0,1]$, $H(t,\cdot)$ is a strict set-contractive mapping. It is easy to show that $H(t,x)$ is uniformly continuous at t for all $x\in\overline{K^R}$. In fact,

$$\begin{aligned}\sup_{u\in\overline{K^R}}\|K_A(g(u)-tNf)-K_A(g(u)-t_0Nf)\| &\le \sup_{u\in\overline{K^R}}\|(g(u)-tNf)-(g(u)-t_0Nf)\|\\ &=\|Nf\|\cdot|t-t_0|,\end{aligned}$$

which implies that $H(t,x)$ is uniformly continuous at t_0 for all $x\in\overline{K^R}$.

We now claim that there exists large enough R such that $u\neq H(t,u)$ for all $t\in[0,1]$, $u\in\partial(\overline{K^R})$. Otherwise, there exist two sequences $\{t_n\}$ and $\{u_n\}$ with $t_n\in[0,1]$, $u_n\in\partial(\overline{K^R})$, and $\|u_n\|\to+\infty$ such that

$$u_n=H(t_n,u_n)=K_A(g(u_n)-t_nNf).$$

Hence,

$$(Au_n,v-u_n)\ge(g(u_n)-t_nNf,v-u_n),\quad\forall v\in K.\tag{2.9}$$

Taking $v=u_n+x_0$ in (2.9), it follows from (2.8) that

$$(Au_n,x_0)\ge(g(u_n),x_0),\tag{2.10}$$

which contradicts the condition (ii). Therefore,

$$i_K(K_{Ag},K^R)=i_K(H(0,\cdot),K^R)=i_K(H(1,\cdot),K^R).\tag{2.11}$$

If $i_K(H(0,\cdot),K^R)\neq0$, then $H(1,\cdot):K\to K$ has a fixed point $u\in K^R$, i.e.,

$$u=H(1,u)=K_A(g(u)-Nf)$$

and so

$$(Au,v-u)\ge(g(u)-Nf,v-u),\quad\forall v\in K.\tag{2.12}$$

Taking $v=u+u_0$ in (2.12), we get

$$(Au,x_0)\ge(g(u),x_0)-N(f,x_0),\tag{2.13}$$

which contradicts the condition (ii) if $\|u\|>q$. Hence $\|u\|\le q$ and so (2.13) implies that

$$-N(f,x_0)\le|(Au,x_0)|+|(gu,x_0)|\le M\|x_0\|+L\|x_0\|,\tag{2.14}$$

which contradicts (2.8). Therefore, $i_K(H(1,\cdot),K^R)=0$.

It follows from (2.7), (2.11) and Lemma 1.1(ii) that $i_K(K_{Ag},K^R\setminus\overline{K^r})=-1$ and thus, there exists a fixed point $u\in K^R\setminus\overline{K^r}$ of K_{Ag}, which is a nonzero solution of the variational inequality (1.1). This completes the proof. □

Theorem 2.2. *Let X be a real reflexive Banach space and K be a closed convex cone of X. Suppose that $A:X\to X^*$ is continuous on finite dimensional subspaces which satisfies condition (2.1) with $A(0)=0$ and $g:K\to X^*$ is β-set-contractive, where $\frac{\beta}{\gamma}<1$. Assume that*

(iii) $\liminf_{\|u_n\|\to+\infty}\frac{(gu_n,u_n)}{\|u_n\|^2}<\gamma$ $(u_n\in K)$;
(iv) *there exist $x_0\in K$ and an open neighbourhood $V(0)$ of zero point such that*

$$(g(u),x_0)>(Au,x_0),\quad\forall u\in K\cap V(0).$$

Then generalized variational inequality (1.1) has a nonzero solution.

Proof. Define a mapping $K_{Ag}:K\to K$ as follows:

$$K_{Ag}(u)=K_A(g(u)),\quad\forall u\in K.\tag{2.15}$$

It is easy to see that K_{Ag} is $\frac{\beta}{\gamma}$-set-contractive by (2.3).

Let $K^r=\{x\in K,\|x\|<r\}$. We shall verify that $i_K(K_{Ag},K^r)=0$ for small enough r and $i_K(K_{Ag},K^R)=1$ for large enough R.

First, we define a mapping $H : [0, 1] \times \overline{K^R} \to K$ as follows:

$$H(t, u) = tK_A(g(u)), \quad \forall (t, u) \in [0, 1] \times \overline{K^R}.$$

Clearly, H is continuous bounded in $[0, 1] \times \overline{K^R}$ and for each $t \in [0, 1]$, $H(t, \cdot)$ is a strict set-contractive mapping. It follows that $H(t, x)$ is uniformly continuous at t for all $x \in \overline{K^R}$.

We claim that there exists large enough $R > 0$ such that $u \neq H(t, u)$ for all $t \in [0, 1]$ and $u \in \partial(K^R)$. Otherwise, there exist two sequences $\{t_n\}$ and $\{u_n\}$ with $t_n \in (0, 1]$, $u_n \in \partial(K^R)$, and $\|u_n\| \to +\infty$ such that

$$u_n = H(t_n, u_n) = t_n K_{Ag}(u_n)$$

or

$$\frac{u_n}{t_n} = K_{Ag}(u_n).$$

Thus, we have

$$\left(A\left(\frac{u_n}{t_n}\right), v - \frac{u_n}{t_n}\right) \geq \left(g(u_n), v - \frac{u_n}{t_n}\right), \quad \forall v \in K. \tag{2.16}$$

Letting $v = 0$, we can obtain from (2.1) and (2.16) that

$$\gamma \leq \frac{(A(\frac{u_n}{t_n}), \frac{u_n}{t_n})}{\|\frac{u_n}{t_n}\|^2} \leq \frac{(g(u_n), \frac{u_n}{t_n})}{\|\frac{u_n}{t_n}\|^2} \leq \frac{(g(u_n), u_n)}{\|u_n\|^2}. \tag{2.17}$$

Hence,

$$\liminf_{\|u_n\| \to +\infty} \frac{(gu_n, u_n)}{\|u_n\|^2} \geq \gamma,$$

which contradicts condition (iii). Therefore,

$$i_K(K_{Ag}, K^R) = i_K(H(1, \cdot), K^R) = i_K(H(0, \cdot), K^R) = i_K(0, K^R) = 1. \tag{2.18}$$

Secondly, we show that $i_K(K_{Ag}, K^r) = 0$ for small enough r. In fact, we can take $r > 0$ such that $\overline{K^r} \subset K \cap V(0)$. By condition (iv), we have

$$(g(u), x_0) > (Au, x_0), \quad \text{as } \|u\| \leq r. \tag{2.19}$$

On the other hand, take N for large enough and $f \in X^*$. Then,

$$0 \leq -N(f, x_0). \tag{2.20}$$

We define a mapping $H : [0, 1] \times \overline{K^r} \to K$ as follows:

$$H(t, u) = K_A(g(u) - tNf), \quad \forall (t, u) \in [0, 1] \times \overline{K^r}.$$

Then $H(t, u)$ is continuous and bounded in $[0, 1] \times \overline{K^R}$. For each $t \in [0, 1]$, $H(t, \cdot)$ is a strict set-contractive mapping and $H(t, x)$ is uniformly continuous at t for all $x \in \overline{K^r}$.

We now claim that there exists small enough $r > 0$ such that $u \neq H(t, u)$ for all $t \in [0, 1]$ and $u \in \partial(\overline{K^r})$. Otherwise, there exist two sequences $\{t_n\}$ and $\{u_n\}$ with $t_n \in [0, 1]$, $u_n \in \partial(\overline{K^r})$, and $\|u_n\| \to 0$ such that

$$u_n = H(t_n, u_n) = K_A(g(u_n) - t_n Nf).$$

Thus,

$$(Au_n, v - u_n) \geq (g(u_n) - t_n Nf, v - u_n), \quad \forall v \in K. \tag{2.21}$$

Taking $v = u_n + x_0$ in (2.21), we have

$$(Au_n, x_0) \geq (g(u_n), x_0), \tag{2.22}$$

which contradicts condition (iv). Therefore,

$$i_K(K_{Ag}, K^r) = i_K(H(0, \cdot), K^r) = i_K(H(1, \cdot), K^r). \tag{2.23}$$

If $i_K(H(1,\cdot), K^r) \neq 0$, then $H(1,\cdot) : K \to K$ has a fixed point $u \in K^r$, i.e.,

$$u = H(1,u) = K_A(g(u) - Nf).$$

It follows that

$$(Au, v-u) \geq (g(u) - Nf, v-u), \quad \forall v \in K. \tag{2.24}$$

Taking $v = u + u_0$ in (2.24), we get

$$(Au, x_0) \geq (g(u), x_0) - N(f, x_0)$$

and so

$$(Au, x_0) \geq (g(u), x_0),$$

which contradicts (2.19). Therefore, $i_K(H(1,\cdot), K^r) = 0$.

Now it follows from (2.18) and (2.23) that

$$i_K(K_{Ag}, K^R \setminus \overline{K^r}) = 1$$

and so there exists a fixed point $u \in K^R \setminus \overline{K^r}$ of K_{Ag}, which is a nonzero solution of the variational inequality (1.1). This completes the proof. $\square$

Remark 2.1. When K_{Ag} is a condensing mapping and K is a star-shaped closed convex set, the fixed point index $i_K(K_{Ag}, U)$ is also well defined. Therefore, Theorems 2.1 and 2.2 also hold when K_{Ag} is a condensing mapping.

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